From Number Sense to Fluency with Basic Combinations

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Presenter

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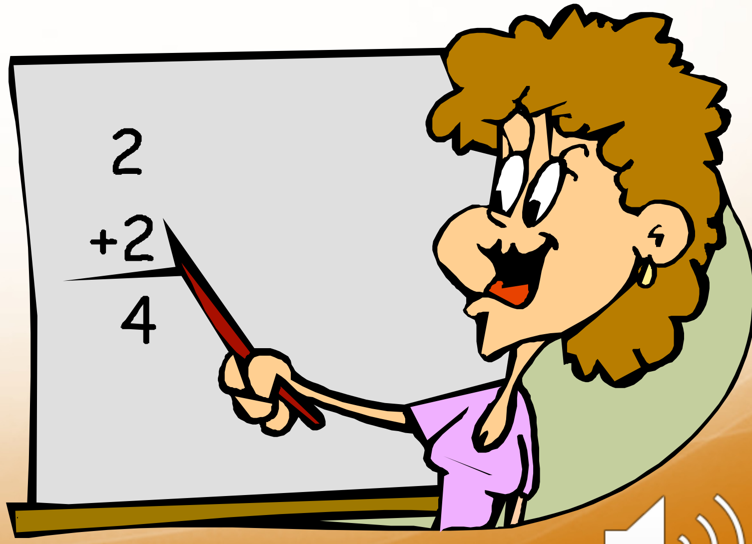
How can I best help students master the basic facts?

The “obvious,” “simple,” and “fast” solution—
endorsed by many educators, policy makers, and parents—
is DRILL, DRILL, and DRILL.

As a parent at a PTO meeting explained:

“If you asked any one of us in the room what 2 plus 2 equals, we would say 4. Why? We were taught in the 1st and 2nd grade the facts by rote memorization.”



Learning the basic sums and differences (or any worthwhile body of knowledge) is much like learning Chinese.

明	brightness
子	child
旦	dawn
森	dense forest or jungle
東	east
炎	extremely hot
火	fire
林	forest
好	good
家	home
月	month
月	moon
安	peace
豕	pig
宀	roof
日	sun
木	tree
女	woman

What are the consequences of learning Chinese characters, the basic sums and differences, or any worthwhile knowledge by rote memorization?

Basically, it makes the task of learning much more difficult than need be.

forgetfulness (lack of retention);

confusion;

inability to apply knowledge (lack of transfer);

negative affect (e.g., math anxiety).

In *Hard Times*, Charles Dickens described
a teacher by the name of Mr. M'Choakumchild
to satirize teachers who focused on
memorizing math by rote
at the expense of all else:

He . . . had taken the bloom off . . .
of mathematics. . . .
He went to work . . . looking at all
the [empty] vessels ranged before
him. . . .
[By using his] boiling store [of
facts to] fill each [of his vessels]
brimful . . . [he killed] outright . . .
[his pupil's curiosity] ... —or
sometimes only maim [it].
(p. 6)

In his 1892 “Talk to Teachers,”
the eminent psychologist William James (1958)
recommended against memorization by rote and
advocated **meaningful memorization**,
which entails **building on existing knowledge:**

“...the art of remembering is the art of *thinking*; ...
when we wish to fix a new thing in a pupil’s [mind],
our conscious effort should not be so much to *impress* and *retain* it
as to *connect* it with something already there.
The connecting *is* the thinking; and,
if we attend clearly to the connection,
the connected thing will ... likely ... remain within recall”
(pp. 101-102).

In the long run, instruction is far more efficient if it focuses on

meaningful memorization

—helping students apply what they know

to discover patterns or relations among facts.

Instead of memorizing by rote each of these 18 Chinese characters separately, consider how much easier and effective it would be to capitalize on the relations among the symbols.

Similarly, meaningfully memorizing
the basic sums/differences,
makes their learning easier
AND—in the long run—produces better results:

retention;

clarity and accuracy;

transfer;

positive affect.

Meaningful memorization
(i.e., using what is known to discover patterns and relations)
is a ***gradual*** **building process.**

In the case of the
basic sums and differences,
the building process *starts with* ***number sense***
constructed in the preschool years.

Meaningful Memorization of the Basic Sums and Differences: A Partial Learning Trajectory

Note: Child needs to master EACH of the first 8 steps to achieve *meaningful memorization of sums to 18 and their related differences (i.e., Step IX)*.

IX. Fluency Adding to 18 & Subtracting

VIII. Deliberate Reasoning Strategies

VII. Counting-based Adding/Subtracting

VI. Number-After Equals 1 More

V. Meaningful Comparison of Verbal Numbers

IV. Number-After and -Before Knowledge

III. Meaningful Comparisons of Collections

II. Meaningful Object Counting

I. Small Number Recognition (Subitizing)

Step I: Small Number Recognition (Subitizing)

Consider Step I—
the **FOUNDATION** for all other steps—
immediate small-number recognition or “subitizing”

(i.e., the ability to immediately recognize and label with a number word the total number in a small collection—without counting).

Research indicates that Kindergartners who cannot subitize up to at least 3—

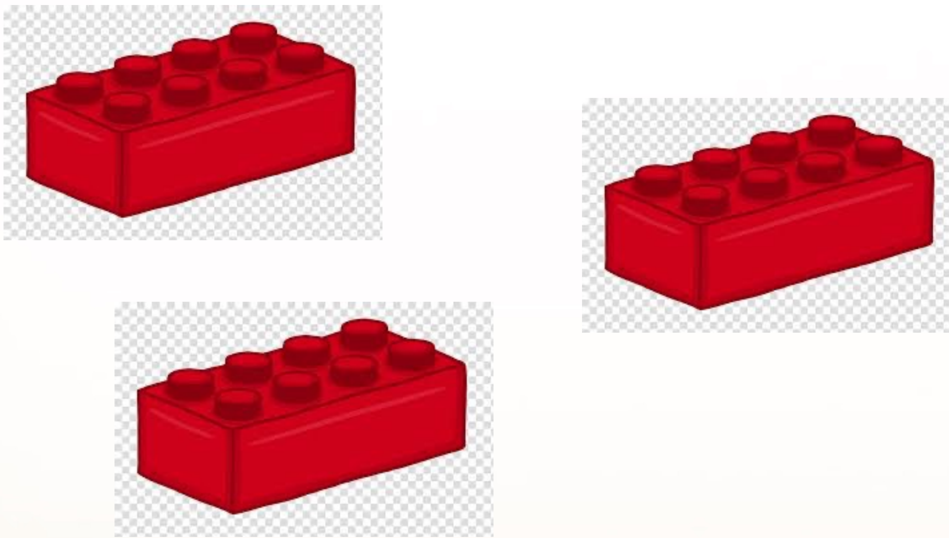
* have little number sense (e.g., understanding of addition),

* low mathematics achievement, and

* are unable to achieve fluency with basic addition facts.

WHY?

Step I: Small Number Recognition (Subitizing)



Step I: Small Number Recognition (Subitizing)

What might a child learn
from repeatedly subitizing such events?

- A basic **understanding of addition** (*adding makes a collection larger*)
- and **subtraction** (*taking away makes it smaller)*.

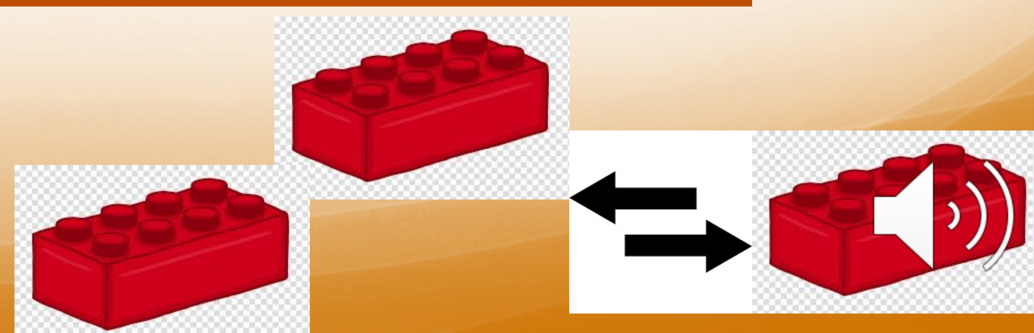
- An **understanding** that the numbers 'one,' 'two,' 'three' and the
- **counting sequence**, in general, **represent increasingly larger collections**.

- Recognition that adding and subtracting are related operations:
  adding and subtracting the same number *undo* each other (**inversion**).

- **Part-whole knowledge** of the small numbers
  (e.g., the part 2 and the part 1 make the whole 3 AND vice versa).

- **Basic facts** (e.g., "Two and one more" makes "three").

- A local understanding of the **successor principle**:
  Adding 1 → results in the next number in the count sequence
  (each successive number is one more than its predecessor).



Key Instructional Guidelines for Promoting Meaningful Learning/Memorization

- Know where a student is on a learning trajectory.

- Chose as an instructional goal
  a level for which a child is developmentally ready
  (often a step one higher than the child's current step).

- Relate new information to what is already known.

- Foster skills, concepts, and mathematical thinking in tandem.

An Example of Key Instructional Guidelines for Promoting Meaningful Learning/Memorization

Step I: Small Number Recognition (Subitizing)

Subitizing of small numbers develops slowly in a stepwise manner—typically along the following learning trajectory:

- Recognition of “1” and “2” (Sub-step 1)

↓

- Recognition of “3” (Sub-step 2)

↓

- Recognition of “4” (Sub-step 3)

Subitizing Activity: *"Make Your Mat the Same Game"*

A tester puts 1 to 4 items on her mat and says: "Make your mat the same as mine."

(In effect, a child is asked to subitize a collection and create a collection that matches the tester's *visible* collection.)

Two-year-old Ki'Shawn

recognized two on the trainer's mat and correctly created a matching collection by putting out 2 on his mat.

Shown *four* items on the tester's mat,
Ki'Shawn simply puts out a bunch—
even though the tester's model collection of four
remains clearly **visible** to him.
For Ki'Shawn, four is literally "many."

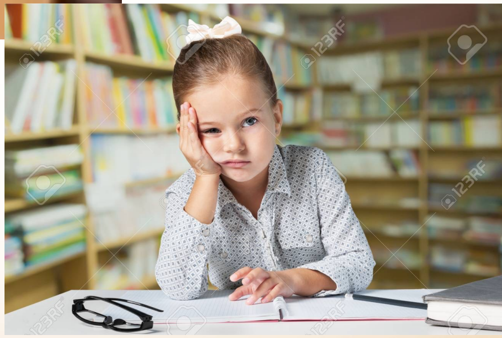
Asked to match 4 again, Ki’Shawn *pushes* the dinosaurs away and then *pushes* his mat into the tester’s mat.

Translation: “This is too hard for me. Let’s do something else.”

Instructional implication? As the child can successfully subitize 2, a teacher should check the next level of difficulty (subitizing 3) *before* introducing subitizing of 4.

“I’m out of here”

Children “melt down” when presented a task too difficult for them.



This is why it is important for a teacher to **build on what a child knows** by

*** identifying where a child is on a learning trajectory and**

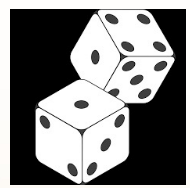
*** focusing on the NEXT step** of the trajectory.

Step I: Small Number Recognition (Subitizing)

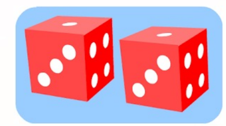
Subitizing small numbers underlies
“**conceptual subitizing**” (*immediate recognition of larger numbers),*
and
and both promote learning of the **“addition doubles.”**

“1 and 1 is 2”

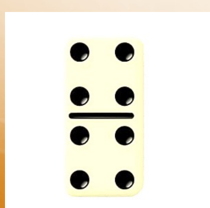
“2 and 2 is 4”



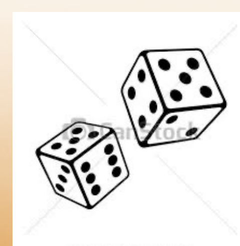
“4 and 1 is 5”

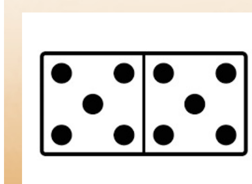


“3 and 3 is 6”

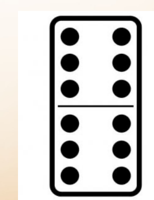


“4 and 4 is 8”





“5 and 5 is 10”



“6 and 6 is 12”

Step IV: Number After

Asked a number-after questions, such as "*When we count, what number comes after six?*"

- Initially, the counting sequence is merely a sing-song— an undifferentiated sound string: "onetwothreefourfive." At this sub-step, children do not even understand the question. [Sub-step 1: *String level of verbal counting*]

- In time, children realize the count sequence is composed of separate number words. At this sublevel, they will understand the question but must use a 'running start' (i.e., count from "one") to determine the answer: "1, 2, 3, 4, 5, 6; seven." [Sub-step 2: *Unbreakable chain level*]

- Kindergartners typically are so familiar with the count sequence, they can immediately enter the sequence at any point and specify the next number: "Seven!" [Sub-step 3: *Breakable chain level*]

Step VIII: Deliberate Reasoning About Adding with 1

Yet, many kindergartners who know number-after relations
such as “after six?” fluently,
do *not* know add-1 facts such as “Six and one.”

For example, while playing a board game,
a kindergarten girl rolled a 6 and 1.
As she did not have enough fingers on one hand to represent 6,
she did not know how to respond.

Marcie who was sitting next to the girl,
leaned over and whispered in ear:
“*That’s an easy one.*
It’s just the number after six.”

Step IX. Fluency Adding or Subtracting with 1

Step VIII: Deliberate Reasoning Strategies Involving 1

Marcie's lessons for fostering fluency with 1 facts.

DO NOT individually DRILL the 19 Add-1 facts and 17 Subtract-1 facts.

1. Instead, BUILD ON PUPIL'S EXISTING KNOWLEDGE OF COUNTING.
   * Help students connect **Adding with 1** to their existing knowledge of **Number-After Relations** to prompt discovery of the ***Number-After Rule*:**
     "The sum of n+1 or 1+n is the number after n in the counting sequence."
   *Help pupils connect **Subtracting 1** to their extant **Number-Before Knowledge** **and invent the *Number-Before Rule*.**

2. Provide opportunities to use the number-after and number-before rules so that they becomes **automatic**.

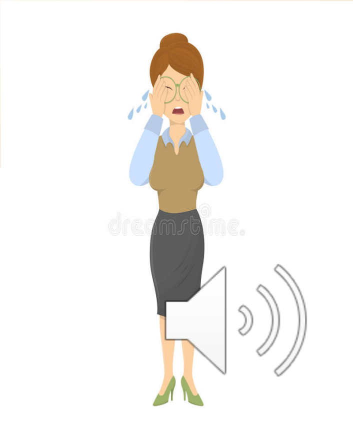
Step IX: Fluency with Subtracting by 2 to 9

The *Common Core State Standards* include as a Grade-1 Goal: learning "**subtraction-as-addition**" **strategy**:

> "Understand subtraction as an unknown-addend problem.
>
> *For example, subtract 10 – 8 by relating it to 8 + ? = 10 (i.e., finding the number that makes 10 when added to 8)."*

The *Common Core State Standards* include as a Grade-2 Goal: **fluency** adding and **subtracting within 20 using mental strategies** such as the "subtraction-as-addition" strategy.

Easier said than done.



A Partial Learning Trajectory of the Sub-steps for Achieving Fluency with Subtracting to 18 – 9 (Step IX)

8 Fluent subtraction-as-addition reasoning

7 Fluency with related addition combinations

6 Deliberate subtraction-as-addition reasoning

5 Complement principle

4 Shared parts-and-whole concept

3 Shared-numbers concept

2 Undoing concept (inversion)

1 Informal basic concept of subtraction

Sub-steps to Step IX: Fluently Subtracting to 18 – 9

Sub-step 2: Undoing concept (inversion)

Addition & subtraction are ***related*** because adding and subtracting the same number of *items* undo each other (e.g., adding 5 to 3 can be undone by subtracting 5 and vice versa).

[For primary pupils, both Sub-steps 1 and 2 typically constitute existing knowledge that develops in the preschool years.]

Helpful to remind children that these ideas also <u>apply to symbolic subtraction</u>.

*** E.g., "Will the answer for 5 – 3 be smaller, equal to, or larger than 5?**

*** E.g., "What is the answer to 5 + 3 = ? and 8 – 3 = ? Or 5 + 3 – 3 = ?"**

Sub-step 1: Informal basic concept of subtraction

Taking items from a collection make the collection smaller.

Sub-sets to Step IX: Fluently Subtracting to 18 – 9

Sub-step 2: Undoing concept (inversion)

Addition & subtraction are ***related*** because adding and subtracting the same number of *items* undo each other (e.g., adding 5 to 3 and then subtracting 5 from 8 makes 3 again)

Why are reminders of these ideas useful?

 # Children may not automatically connect informal knowledge to symbolic representations.

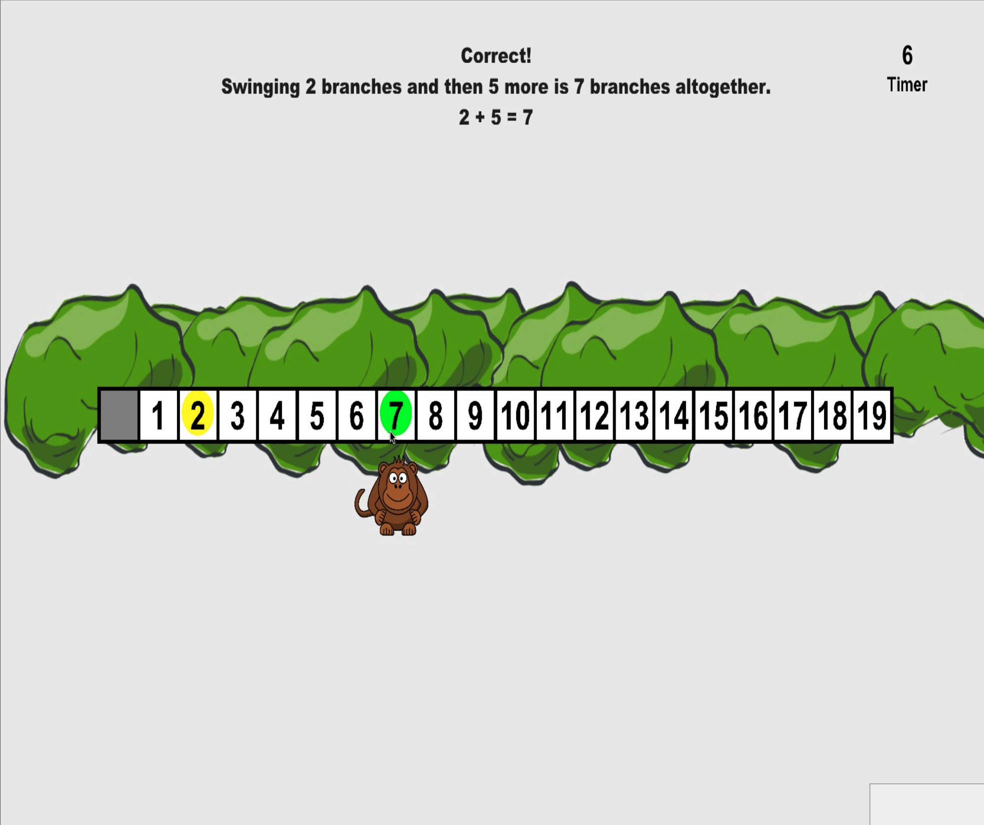
*# An undoing concept underscores addition and subtraction are **related**, not separate, operations.*

 # An undoing concept is helpful in understanding Sub-steps 3 and 8.

 # Math textbooks typically do not address the undoing concept.

Sub-step 1: Informal basic concept of subtraction

Taking items from an original collection make the collection smaller.



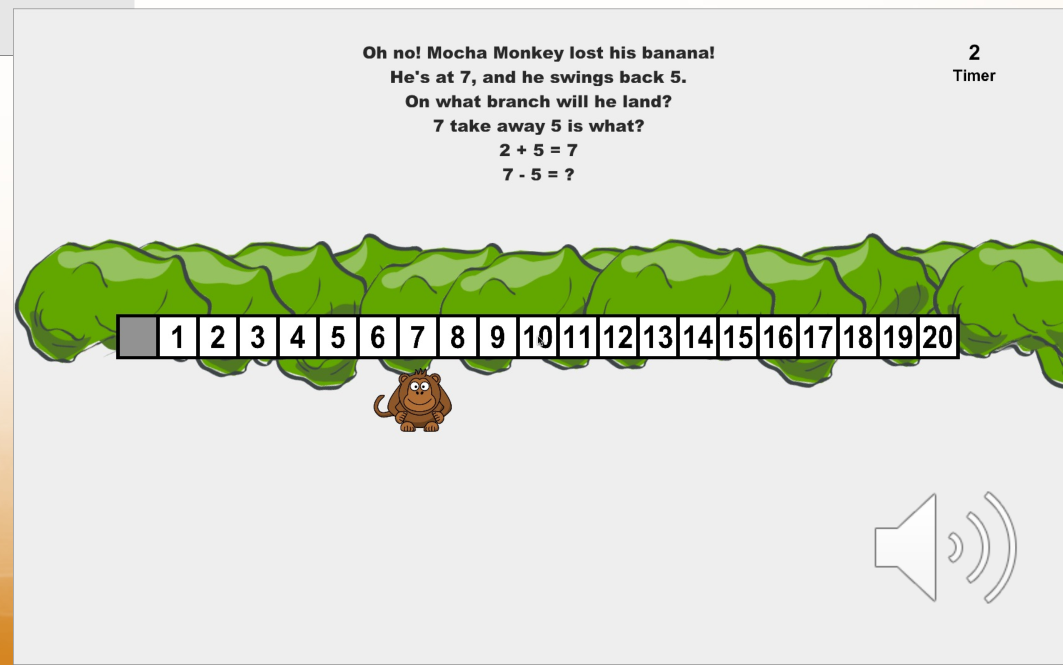
Underscoring the *undoing concept* (inversion)

First, ask child to calculate the answer to, for example, 2 + 5 = ?

And provide feedback.

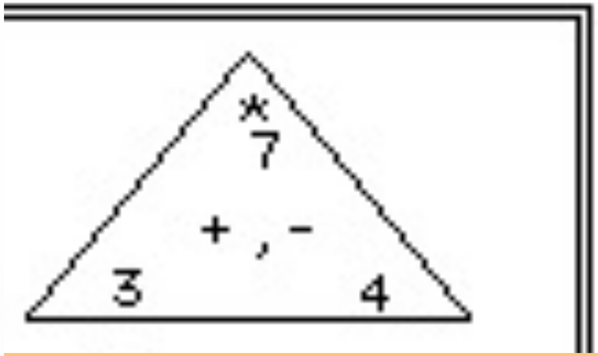
Then ask child to answer 7 – 5 = ?

(i.e., undo the addition of 5)



Sub-steps to Step IX: Fluently Subtracting to 18 – 9

Fact Triangles



Can serve to illustrate the undoing concept (Sub-step 2)

This is seldom done in textbooks.

University of Chicago School Mathematics Project. (2005). *Everyday mathematics teacher's lesson guide* (Vol. 1).

Sub-step 2: Undoing concept (inversion)

Sub-step 1: Informal basic concept of subtraction

Sub-steps to Step IX: Fluently Subtracting to 18 – 9

Sub-step 4: Shared parts-and-whole concept ("family of part-whole complements")

A "family of part-whole complements," such as 5 + 3 = 8, 3 + 5 = 8, 8 – 3 = 5, 8 – 5 = 3, shares the same **whole** 8 and **parts 3 and 5**.

+ explicit labeling of parts and the whole

Sub-step 3: Shared-numbers concept ("family of addition-subtraction complements")

A "family of addition and subtraction complements," such as 5 + 3 = 8, 3 + 5 = 8, 8 – 3 = 5, 8 – 5 = 3, all share the same three numbers: 3, 5, 8.

Sub-steps to Step IX: Fluently Subtracting to 18 – 9

The Complement Principle (Sub-step 5) serves as the rationale or conceptual basis for the subtraction-as-addition strategy (Sub-step 6).

The Complement Principle is often not addressed by textbooks.

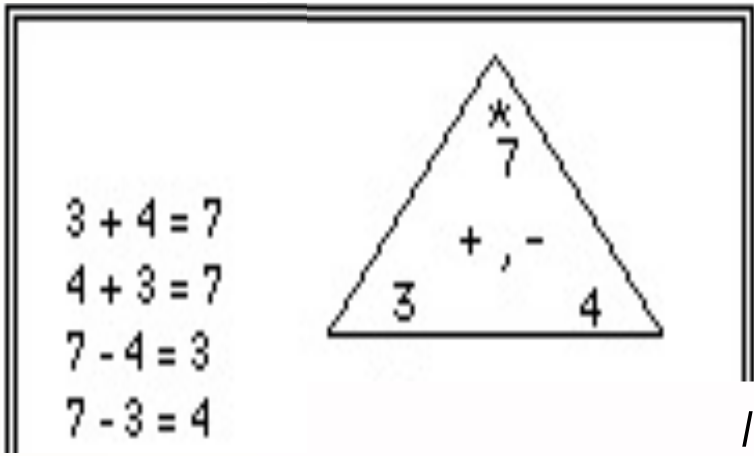
If addressed by a textbook, the complement principle is often *not* related to the shared parts-whole concept (Sub-step 4)

Sub-step 5: Complement principle

If Part a + Part b = Whole c,
then the Whole c – Part b = Part a
(e.g., if the parts 5 and 3 make the whole 8,
then the whole 8 minus the part 3 leaves the part 5)

Sub-step 4: Shared parts-and-whole concept
= "family of part-whole complements"

Sub-steps to Step IX: Fluently Subtracting to 18 – 9



Students can use *fact triangles* and *the undoing concept* to **create** families of addition-subtraction complements & illustrate the shared-numbers concept (Sub-step 3).

If used with explicit part-whole labeling, fact triangles can serve to illustrate a shared parts-whole concept (Sub-step 4) or the complement principle (Sub-step 5), (both of which are seldom done in textbooks).

Sub-step 5: Complement principle

Sub-step 4: Shared parts-and-whole concept = "family of part-whole complements"

Sub-step 3: Shared-numbers concept = "family of addition-subtraction complements"

Sub-step 2: Undoing concept (inversion)

Sub-steps to Step IX: Fluently Subtracting to 18 – 9

Primary-level textbooks commonly introduce
a subtraction-as-addition strategy (Sub-step 6).

They are inconsistent in relating this strategy to the
complement principle (Sub-step 5).

It might help to do so
(e.g., What family member is related to 8 – 3 = ?
and can be used to solve the unknown difference?).

Sub-step 6: Deliberate subtraction-as-addition reasoning
Use a known sum to determine an unknown difference

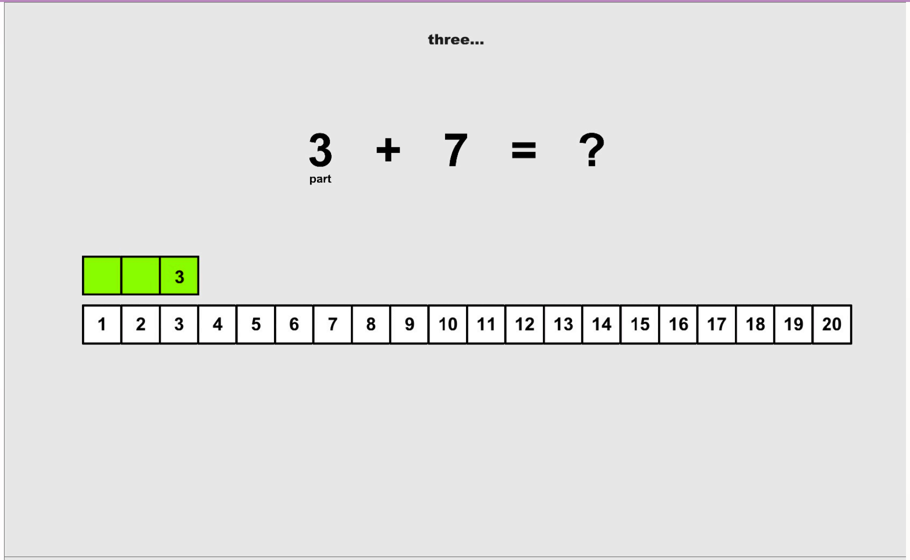
Sub-step 5: Complement principle

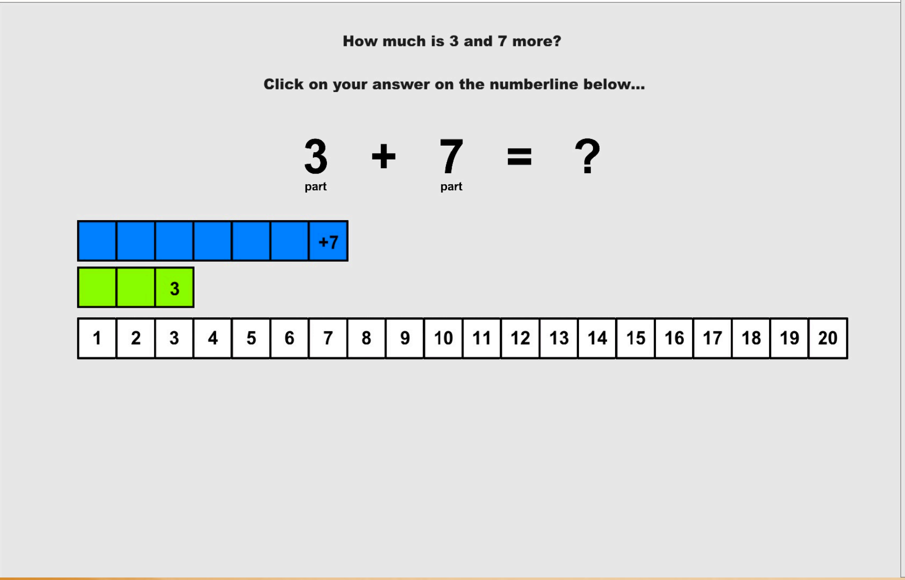
Sub-steps to Step IX: Fluently Subtracting to 18 – 9

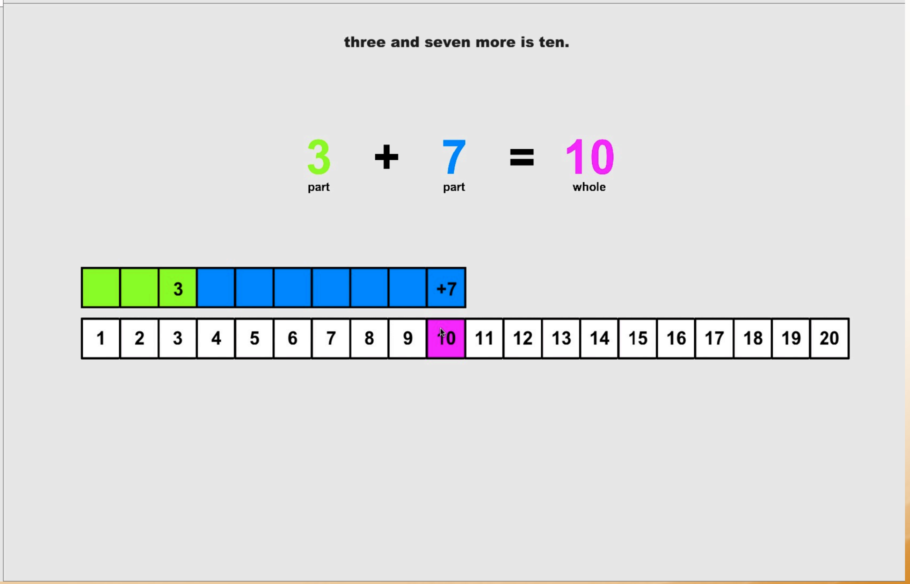
Helping Pupils Discover the Subtraction-as-Addition Strategy

Introduce a helper (related addition) item.
For example, how much is three and seven more altogether?

And provide feedback.







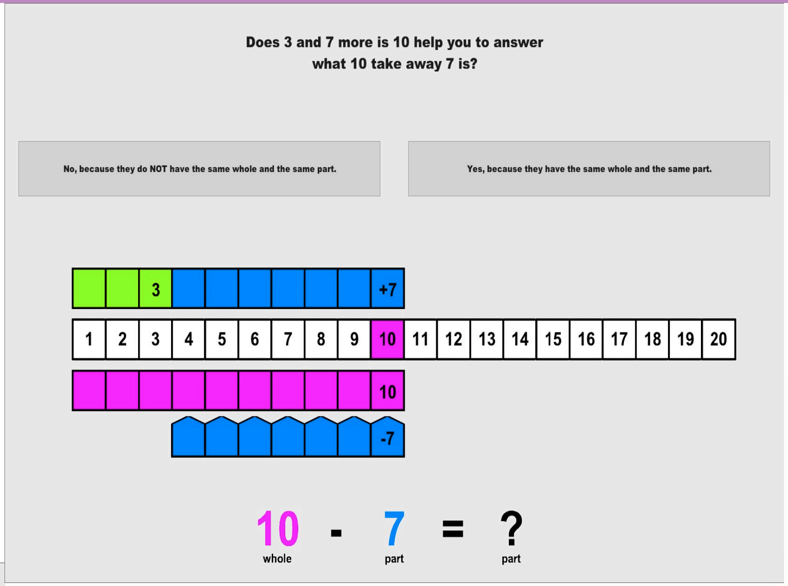
Sub-step 6: Deliberate subtraction-as-addition reasoning

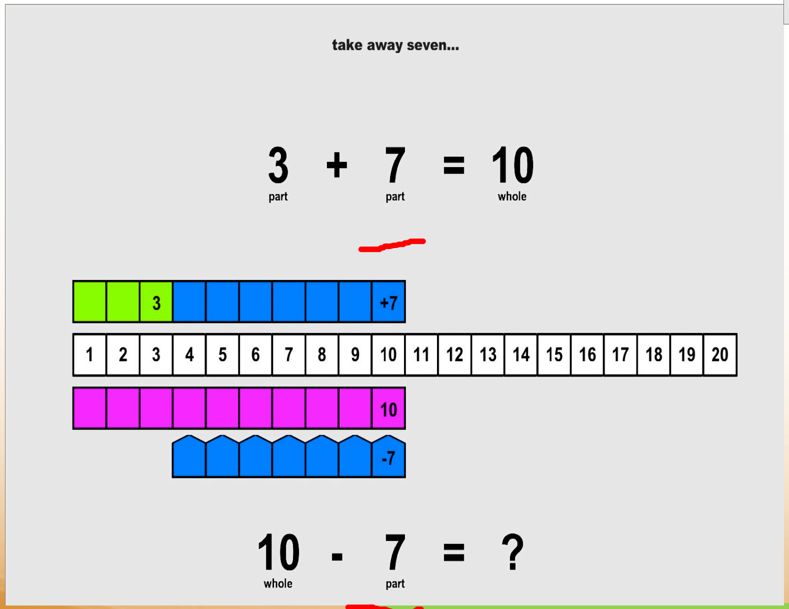
Sub-step 5: Complement principle

Sub-steps to Step IX: Fluently Subtracting to 18 – 9

Then, encourage students to consider whether the addition fact may be useful in solving a new subtraction problem.

For example, can ‘3 and 7 more is 10’ help solve ‘10 take away 7’?





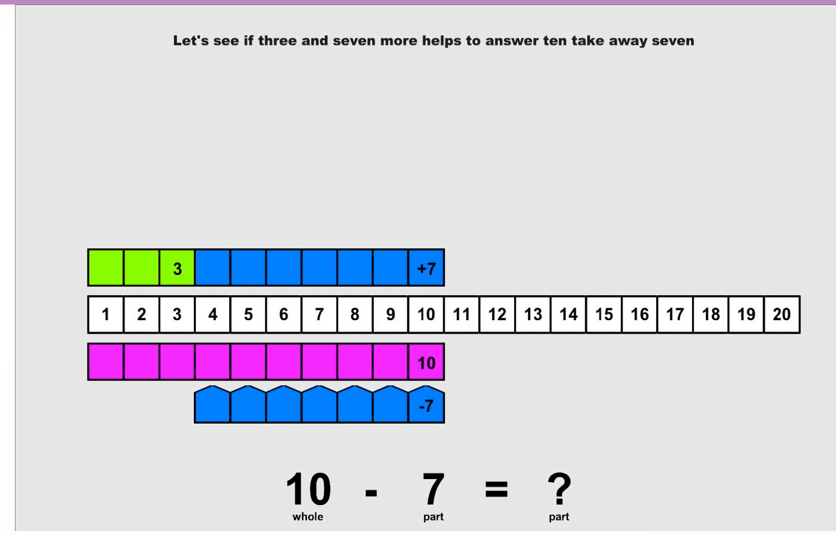
Note that the part 7 is in the same position for both the addition and subtraction equations to better underscore they are from the same complement family.

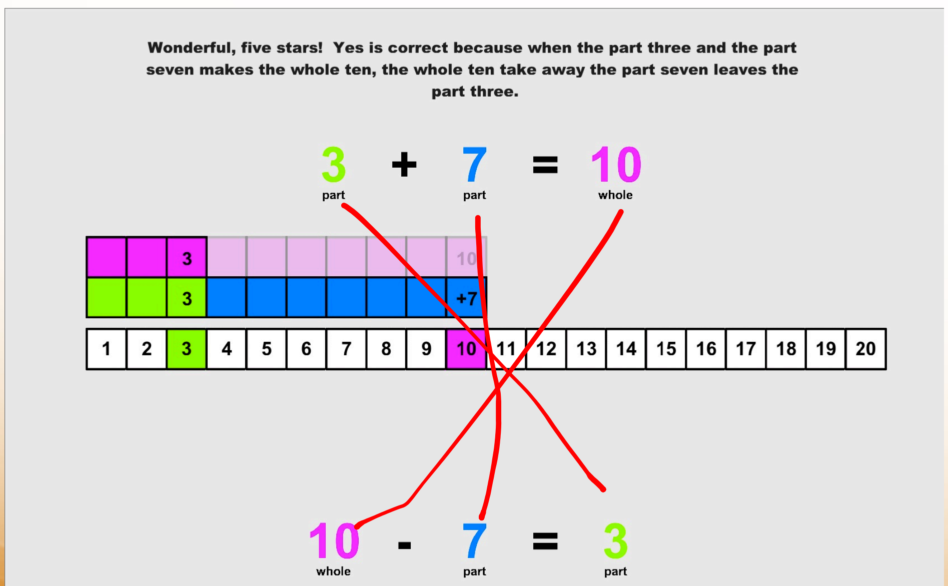
Sub-step 6: Deliberate subtraction-as-addition reasoning

Sub-step 5: Complement principle

Sub-steps to Step IX: Fluently Subtracting to 18 – 9

Next, encourage students to calculate the difference to check their prediction.





Note that the feedback explicitly summarizes the <u>complement principle</u>.

Note that the corresponding parts and wholes in the addition and subtraction equations have the same label and the same color to underscore the <u>shared-parts-and-whole concept</u>.

Sub-step 6: Deliberate subtraction-as-addition reasoning

Sub-step 5: Complement principle

Step IX: Fluently Subtracting to 18 – 9

Sub-step 6
With practice, children may *readily* apply subtraction-as-addition reasoning

(e.g., for 8 - 5 = ? QUICKLY think:
What part plus the part 5 makes the whole 8?).

Sub-steps 7 & 8
With more practice, children can non-consciously and automatically use a stored triad to efficiently retrieve sums and differences

(e.g., access the 3-5-8 triad to fluently determine
that the sum of 3 + 5 is 8 or
the difference of 8 – 5 is 3).

Sub-step 8: Fluent subtraction-as-addition reasoning

Sub-step 7: Fluency with related addition combinations

Sub-step 6: Deliberate subtraction-as-addition reasoning

Step IX: Fluently Subtracting to 18 – 9

Note. Practice, *including* timed practice, is a **useful** instructional tool—
IF used properly and carefully.

Specifically, practice and timed practice be useful
IF done in the service of ***meaningful* memorization**
(as opposed to memorization by rote).

To achieve Sub-steps 7 & 8
Timed practiced can serve to automatize
the basic sums and the reasoning process
necessary to implement subtraction-as-addition reasoning fluently.

To achieve Sub-step 6
Practice can provide a basis for discovering and appropriately applying
deliberate subtraction-as-addition reasoning.

To achieve Sub-steps 2 to 5
Practice informally determining sums and differences can provide a
basis for discovering and reinforcing (conceptual) prerequisites for the
meaningful learning the subtraction-as-addition strategy.

Key Instructional Guidelines for Promoting Meaningful Learning/Memorization

- Know where a student is on a learning trajectory.
- Chose as a goal a level for which a child is developmentally ready (i.e., often a level one higher than the child's current level).
- Relate new information to what is already known.
- Foster skills, concepts, and mathematical thinking in tandem.

References

The following provide more information on how the **learning trajectory** depicted in this presentation—how early number sense supports achieving fluency with basic addition combinations—and the **assessment/instructional games** for specific levels:

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Thank you for listening.

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